Symmetrical Parametrizations of the Lepton Mixing Matrix

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Advantages of the original symmetrical form of the parametrization of the lepton mixing matrix are discussed. It provides a conceptually more transparent description of neutrino oscillations and lepton number violating processes like neutrinoless double beta decay, clarifying the significance of Dirac and Majorana phases. It is also ideal for parametrizing scenarios with light sterile neutrinos.

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I. INTRODUCTION

Since the historic discovery of neutrino oscillations, massive neutrinos currently provide the most direct and testable evidence for physics beyond the Standard Model (SM) of particle physics. At low energies, nine (seven) parameters must be determined depending on whether neutrinos are Majorana (Dirac) particles [1][1]. Here we tacitly assume the former, more general, and theoretically preferred case. Parametrizing the lepton mixing matrix [2, 3] in a convenient and intuitive manner is very helpful for data handling and interpretation of the physics, such as neutrino oscillation searches in upcoming long-baseline experiments [4, 5] or searches for neutrinoless double beta decay [6, 7]. While the former are sensitive to the Dirac phase, Majorana phases [1, 8–11] are crucial to describe the latter.

The Particle Date Group (PDG) has adopted a parametrization of the lepton mixing matrix in which it is a product of three consecutive rotations multiplied with a diagonal phase matrix P containing the Majorana phases [12]. The mixing matrix can be written as

$$U = R_{23}(\theta_{23}; 0)\, R_{13}(\theta_{13}; \delta)\, R_{12}(\theta_{12}; 0)\, P\,, \tag{1}$$

where $R_{ij}(\theta; \varphi)$ is a rotation around the ij-axis, e.g.

$$R_{13}(\theta_{13}; \delta) = \begin{pmatrix} \cos\theta_{13} & 0 & \sin\theta_{13}\, e^{-i\delta} \\ 0 & 1 & 0 \\ -\sin\theta_{13}\, e^{i\delta} & 0 & \cos\theta_{13} \end{pmatrix}. \tag{2}$$

The position of the Dirac phase δ is the convention chosen by the PDG. The two Majorana phases, denoted here α and β, are usually put inside P, to the right of the mixing matrix: $P = \mathrm{diag}(e^{i\alpha}, e^{i\beta}, 1)$. However there is no consensus notation yet in what concerns the parametrization of these phases, neither for their names (e.g. ϕ_1 and ϕ_2 or φ_1 and φ_2 or σ and ρ, sometimes with a minus sign, sometimes divided by two, etc.), nor for their position within the matrix ($P = \mathrm{diag}(1, e^{i\alpha}, e^{i\beta})$, or $P = \mathrm{diag}(e^{i\alpha}, e^{i\beta}, 1)$, etc.). The mixing matrix Eq. (1) is explicitly given by

$$U \equiv \tilde{U}\, P = \begin{pmatrix} c_{12}c_{13}\, e^{i\alpha} & s_{12}c_{13}\, e^{i\beta} & s_{13}\, e^{-i\delta} \\ -(s_{12}c_{23} + c_{12}s_{23}s_{13}\, e^{i\delta})\, e^{i\alpha} & (c_{12}c_{23} - s_{12}s_{23}s_{13}\, e^{i\delta})\, e^{i\beta} & s_{23}c_{13} \\ (s_{12}s_{23} - c_{12}c_{23}s_{13}\, e^{i\delta})\, e^{i\alpha} & -(c_{12}s_{23} + s_{12}c_{23}s_{13}\, e^{i\delta})\, e^{i\beta} & c_{23}c_{13} \end{pmatrix}. \tag{3}$$

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[1] Many more parameters exist if neutrino masses arise from type-I seesaw, the parametrization in Ref. [1] covers all seesaw cases.

The elements $|U_{e1}|$ and $|U_{\mu 3}|$ are known with good accuracy, thus the two "large" angles θ_{12} and θ_{23} are well-determined by solar and atmospheric neutrino oscillation data. There is also recent evidence for a nonzero value of the "small" element $|U_{e3}|$ from the T2K Collaboration [13] and the global neutrino oscillation data sample [14, 15]. While there are 8 equivalent ways to parametrize the mixing matrix [16], the above order of rotation is useful in the sense that experimentally the straightforwardly measurable elements $|U_{e1}|$, $|U_{e3}|$ and $|U_{\mu 3}|$ allow to directly extract the angles in the three–neutrino lepton mixing matrix [2]. In contrast, the other equivalent parametrizations do not share this property.

II. A MORE CONVENIENT PARAMETRIZATION OF THE MIXING MATRIX

The above form U is nothing but a re-writing of the "symmetrical" form K proposed in Ref. [1] (apart from factor ordering, which was left unspecified in the original paper). Here we would like to argue in favor of the conceptual advantages of the original "symmetrical" presentation of the lepton mixing matrix [1]. For the case of three neutrinos it is given as:

$$K = \omega_{23}(\theta_{23};\phi_{23})\omega_{13}(\theta_{13};\phi_{13})\omega_{12}(\theta_{12};\phi_{12})\,, \tag{4}$$

where each of the ω's is effectively 2×2, characterized by an angle and a CP phase, e.g.

$$\omega_{13} = \begin{pmatrix} c_{13} & 0 & e^{-i\phi_{13}}s_{13} \\ 0 & 1 & 0 \\ -e^{i\phi_{13}}s_{13} & 0 & c_{13} \end{pmatrix}.$$

Explicitly, the symmetrical parametrization of the lepton mixing matrix, K can be written as:

$$K = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13}e^{-i\phi_{12}} & s_{13}e^{-i\phi_{13}} \\ -s_{12}c_{23}e^{i\phi_{12}} - c_{12}s_{13}s_{23}e^{-i(\phi_{23}-\phi_{13})} & c_{12}c_{23} - s_{12}s_{13}s_{23}e^{-i(\phi_{12}+\phi_{23}-\phi_{13})} & c_{13}s_{23}e^{-i\phi_{23}} \\ s_{12}s_{23}e^{i(\phi_{12}+\phi_{23})} - c_{12}s_{13}c_{23}e^{i\phi_{13}} & -c_{12}s_{23}e^{i\phi_{23}} - s_{12}s_{13}c_{23}e^{-i(\phi_{12}-\phi_{13})} & c_{13}c_{23} \end{pmatrix}. \tag{5}$$

Here all three CP violating phases are physical [9]: ϕ_{12}, ϕ_{23} and ϕ_{13}. Even though the parametrization is fully "symmetric" there is a basic difference between Dirac and Majorana phases. In order to understand this let us use the identity [1]

$$P^{-1}\,K\,P = \omega_{23}(\theta_{23};\phi_{23}-\beta)\;\omega_{13}(\theta_{13};\phi_{13}-\alpha)\;\omega_{12}(\theta_{12};\phi_{12}+\beta-\alpha)\,, \tag{6}$$

which allows us, up to unphysical phases, to identify the Dirac phase as

$$\delta \leftrightarrow \phi_{13}-\phi_{12}-\phi_{23}\,, \tag{7}$$

This formula relates the Dirac phase, which denotes the phase responsible for CP violation in neutrino oscillations in the PDG parametrization, with the equivalent phase in the symmetrical representation. Note that it obeys field rephasing invariance, as it should. Moreover, in contrast to the PDG description, in the symmetrical form CP violation in neutrino oscillations is immediately recognized as a three-generation phenomenon involving the phases of all three generations[3], an important conceptual advantage.

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[2] See Ref. [17] for a recent application of the other possible parametrizations, and [18] for a rare use of the symmetrical parametrization we will study in this work.
[3] In this sense the Dirac phase has an intrinsic geometric meaning, like the curl of two vectors or the area of a triangle.

Recall that the neutrino oscillation probability for a $\nu_\alpha \to \nu_\beta$ flavor transition is given by

$$P(\nu_\alpha \to \nu_\beta) = \left| \sum_j U^*_{\alpha j} U_{\beta j}\, \mathrm{e}^{-i\frac{m_j^2}{2E}L} \right|^2 = \delta_{\alpha\beta} - 4 \sum_{i>j} \Re\left\{ U^*_{\alpha i} U_{\alpha j} U_{\beta i} U^*_{\beta j} \right\} \sin^2\left(\frac{\Delta m^2_{ij}}{4E} L \right)$$
$$+2 \sum_{i>j} \Im\left\{ U^*_{\alpha i} U_{\alpha j} U_{\beta i} U^*_{\beta j} \right\} \sin\left(\frac{\Delta m^2_{ij}}{2E} L \right),$$

where E is the neutrino energy, L is the distance traveled by the neutrino, and $\Delta m^2_{ij} \equiv m_i^2 - m_j^2$ (m_i being positive mass eigenvalues) are the neutrino mass-squared differences. Here $\Re$ and $\Im$ denote real and imaginary parts. For three families there is only one independent imaginary part $\Im\left\{ U^*_{\alpha i} U_{\alpha j} U_{\beta i} U^*_{\beta j} \right\}$, which is responsible for CP violation in neutrino oscillations. Comparing this invariant with the PDG and symmetrical parametrizations gives

$$J_{\mathrm{CP}} \;=\; \Im\left\{ U^*_{e1}\, U^*_{\mu 3}\, U_{e3}\, U_{\mu 1} \right\} = \begin{cases} \frac{1}{8} \sin 2\theta_{12}\, \sin 2\theta_{23}\, \sin 2\theta_{13}\, \cos\theta_{13}\, \sin\delta & \text{(PDG)}, \\ \frac{1}{8} \sin 2\theta_{12}\, \sin 2\theta_{23}\, \sin 2\theta_{13}\, \cos\theta_{13}\, \sin(\phi_{13} - \phi_{12} - \phi_{23}) & \text{(symmetrical)}, \end{cases}$$

and shows the same result as Eq. (7).

A. Application to Lepton Number Violating Phenomena

As well known, there are two conceptually different kinds of CP violating phenomena [1]. In the language of the PDG parametrization, one is associated to the "Dirac phase" δ and is the exact analogue to the CP phase in the quark mixing matrix, responsible for the area of the Cabibbo-Kobayashi-Maskawa (CKM) unitarity triangle; while the other one is associated to the two "Majorana phases" α and β, which do not show up in neutrino oscillations [8–11] but do affect lepton number violating amplitudes.

In what follows we will briefly discuss the role of Majorana phases in determining the rates characterizing neutrinoless double beta decay and neutrino-anti-neutrino oscillations [9].

A suitable parametrization of Majorana phases plays a very important role in interpreting the effective mass parameter characterizing the amplitude for neutrinoless double beta decay (see Ref. [7] for a recent review). Its explicit form reads

$$\langle m \rangle = \left| \sum_j U^2_{ej}\, m_j \right| = \begin{cases} \left| c^2_{12} c^2_{13}\, m_1\, \mathrm{e}^{2i\alpha} + s^2_{12} c^2_{13}\, m_2\, \mathrm{e}^{2i\beta} + s^2_{13}\, m_3\, \mathrm{e}^{2i\delta} \right| & \text{(PDG)}, \\ \left| c^2_{12} c^2_{13}\, m_1 + s^2_{12} c^2_{13}\, m_2\, \mathrm{e}^{2i\phi_{12}} + s^2_{13}\, m_3\, \mathrm{e}^{2i\phi_{13}} \right| & \text{(symmetrical)}. \end{cases} \tag{8}$$

Only the two Majorana phases should appear in $\langle m \rangle$ [9]. However this is not at all clear in the PDG presentation. In contrast, the symmetrical parametrization provides a manifestly transparent description in which only the two Majorana phases appear in $\langle m \rangle$, as it should. Currently nuclear matrix element uncertainties prevent the extraction of Majorana phases from neutrinoless double beta decay. However, should these be circumvented and should the determination of the Majorana phases become an issue, then the symmetrical parametrization will surely be preferred over the PDG one.

It has long been known that the lepton mixing matrix characterizing the charged current interaction of Majorana neutrinos in gauge theories may have complex entries that conserve CP [19]. These special CP conserving situations are associated with Wolfenstein's CP-signs [20], when neutrino mass states are CP eigenstates with CP parity $\eta_{\mathrm{CP}} = \pm i$. There are four possible inequivalent sign configurations in the sum in Eq. (8): $(+++)$, $(+--)$, $(+-+)$, and $(++-)$. These are in correspondence to special values of the Majorana phases, namely $\phi_{12} = \phi_{13} = 0$, $\phi_{12} = \phi_{13} = \pi/2$, $\phi_{13} = \pi/2$ with $\phi_{12} = 0$, and $\phi_{13} = 0$ with $\phi_{12} = \pi/2$, respectively. Majorana phases would also show up in processes analogous to neutrinoless double beta decay, such as decays like $K^+ \to \pi^-\, \mu^+ \mu^+$, whose amplitude would be proportional to $\sum U^2_{\mu i}\, m_i$, and have extremely low branching ratios, see Ref. [7] and references therein.

Let us now comment on neutrino–anti-neutrino oscillations. A Gedankenexperiment looking for anti-neutrinos $\bar{\nu}_\beta$ in a beam of neutrinos ν_α has been suggested in Ref. [9] in order to clarify the physical nature of Majorana phases at the two-generation level. In the three-generation case the probability for such a process is given as

$$P(\nu_\alpha \to \bar{\nu}_\beta) \;=\; \frac{1}{E^2} \left| \sum_j U_{\alpha j} U_{\beta j}\, m_j\, \mathrm{e}^{-iE_j t} \right|^2 \;=\; \frac{1}{E^2} \left| \sum_{i,j} U_{\alpha j} U_{\beta j} U^*_{\alpha i} U^*_{\beta i}\, m_i m_j\, \mathrm{e}^{-i(E_j - E_i)t} \right| . \tag{9}$$

leading to complicated transition probability expressions, which will not be explicitly given here. The least unrealistic channel is represented by ν_e to $\bar{\nu}_e$ transitions, because the transition probabilities go with the ratio of mass over energy squared, and electron neutrinos can be produced with much lower energy than the other flavors. For three families, $P(\nu_e \to \bar{\nu}_e)$ depends only on the Majorana phases ϕ_{12} and ϕ_{13} for the symmetrical parametrization, whereas the PDG case leads to a dependence on all three phases. Note the analogy to the effective mass discussed above.

B. Application to seesaw and sterile neutrinos

We now turn to the lepton mixing matrix characterizing models containing gauge singlets such as seesaw models. Their most general form was presented within the symmetrical parametrization in Ref. [1], covering seesaw schemes of all types, type-I, type-II, and type-III. Since it applies to an arbitrary number m of non-doublet leptons (singlets in type-I and II, triplets in type-III) this parametrization also covers low-scale in addition to high-scale seesaw schemes. To a good approximation, in the standard high-scale seesaw case neutrino oscillations are well-described by the simplest unitary form of the lepton mixing matrix used above in Eqs. (1) or (4). In contrast, for the low-scale seesaw schemes [21–24] neutrino oscillations involve only a unitarity-violating truncation of the full mixing matrix, see, for example, Refs. [25, 26]. In both cases one has an "effective" neutrino oscillation description with $m = 0$, i.e. the extra neutral states are too heavy to take part in the oscillation phenomena. Since these possibilities have already been widely discussed in the literature here we will focus on the alternative possibility that singlets are light enough to participate in oscillations, in the simple case of $m = 1$, i.e. one "sterile" neutrino plus three active ones. This possibility has recently re-gained attention [27]. In terms of the mixing matrix, a useful order of rotation is 34-24-14-23-13-12. We have therefore

$$U = \omega_{34}(\theta_{34}; 0)\, \omega_{24}(\theta_{24}, \delta_{24})\, \omega_{14}(\theta_{14}; \delta_{14})\, \omega_{23}(\theta_{23}; 0)\, \omega_{13}(\theta_{13} : \delta_{13})\, \omega_{12}(\theta_{12}; 0)\, P \,, \tag{10}$$

with $P = \mathrm{diag}(\mathrm{e}^{i\alpha}, \mathrm{e}^{i\beta}, \mathrm{e}^{i\gamma})$ in the sense of the PDG description, or in the symmetrical form:

$$K = \omega_{34}(\theta_{34}; \phi_{34})\, \omega_{24}(\theta_{24}, \phi_{24})\, \omega_{14}(\theta_{14}; \phi_{14})\, \omega_{23}(\theta_{23}; \phi_{23})\, \omega_{13}(\theta_{13}; \phi_{13})\, \omega_{12}(\theta_{12}; \phi_{12}) \,. \tag{11}$$

First note that the number of rotations for 3 + 1 neutrino types (six) is exactly the number of phases (3 Dirac and 3 Majorana), a characteristic feature of the symmetrical parametrization [1].

Consider first the effective mass characterizing neutrinoless double beta decay in this case, which is given as

$$\langle m \rangle = \begin{cases} \left| c_{12}^2 c_{13}^2 c_{14}^2\, m_1\, \mathrm{e}^{2i\alpha} + s_{12}^2 c_{13}^2 c_{14}^2\, m_2\, \mathrm{e}^{2i\beta} + s_{13}^2 c_{14}^2\, m_3\, \mathrm{e}^{2i(\gamma - \delta_{13})} + s_{14}^2\, m_4\, \mathrm{e}^{-2i\delta_{14}} \right| & \text{(PDG)} \,, \\ \left| c_{12}^2 c_{13}^2 c_{14}^2\, m_1 + s_{12}^2 c_{13}^2 c_{14}^2\, m_2\, \mathrm{e}^{2i\phi_{12}} + s_{13}^2 c_{14}^2\, m_3\, \mathrm{e}^{2i\phi_{13}} + s_{14}^2\, m_4\, \mathrm{e}^{2i\phi_{14}} \right| & \text{(symmetrical)} \,, \end{cases}$$

Obviously the symmetrical parametrization has advantages over the one of PDG. Indeed, again, the CP conserving cases corresponding to different CP-signs are obtained by choosing the three Majorana phases $\phi_{12,13,14}$ to be $\pi/2$ or zero.

In what regards oscillations, the three Dirac phases in the PDG description are δ_{13}, δ_{14} and δ_{24}. While there are nine [28] independent $J^{ij}_{\alpha\beta} = \Im \left\{ U^*_{\alpha i}\, U^*_{\beta j}\, U_{\alpha j}\, U_{\beta i} \right\}$, only three independent CP asymmetries $P(\nu_\alpha \to \nu_\beta) - P(\nu_\beta \to \nu_\alpha)$ exist. In the symmetrical parametrization, the relevant independent phase combinations appearing in oscillation probabilities are

$$\begin{aligned} I_{123} &= \phi_{12} + \phi_{23} - \phi_{13} \,, \\ I_{124} &= \phi_{12} + \phi_{24} - \phi_{14} \,, \\ I_{134} &= \phi_{13} + \phi_{34} - \phi_{14} \,. \end{aligned} \tag{12}$$

Each of the phase combinations I_{ijk}, with $i<j<k$, is seen to span three generations, as necessary for the existence of CP violation in neutrino oscillations. Note that there is a fourth possible combination, I_{234}. However, since

$$I_{123}+I_{134}-I_{124}=I_{234} \tag{13}$$

holds, this fourth invariant is not independent. This condition actually implies that I_{ijk} is a 2-cocycle [29]. This is true for an arbitrary number N_s of additional sterile neutrinos: noting that the $\frac{1}{2}\,N_s\,(N_s-1)$ rotations between sterile neutrinos are unphysical, it is easy to see that for N massive neutrinos, including $N_s=N-3$ sterile neutrinos, there are $N-1=N_s+2$ Majorana phases and $2\,N-5=2\,N_s+1$ Dirac phases. Each massless neutrinos results in one Majorana phase less. The total number of $3\,(N-2)=3\,(N_s+1)$ phases corresponds exactly to the number of physical rotations, which is $\frac{1}{2}\,N\,(N-1)-\frac{1}{2}\,N_s\,(N_s-1)=3\,(N-2)$. The symmetrical parametrization is therefore tailor-made also for concisely describing the phenomenology of sterile neutrinos.

It has been argued that current neutrino data might imply in fact that 2 sterile neutrinos are present [30]. A possible order of the nine physical rotations is

$$\begin{aligned} K=\omega_{25}(\theta_{25};\phi_{25})\,\omega_{34}(\theta_{34};\phi_{34})\,\omega_{35}(\theta_{35};\phi_{35})\,\omega_{24}(\theta_{24},\phi_{24})\\ \omega_{23}(\theta_{23};\phi_{23})\,\omega_{15}(\theta_{15};\phi_{15})\,\omega_{14}(\theta_{14};\phi_{14})\,\omega_{13}(\theta_{13};\phi_{13})\,\omega_{12}(\theta_{12};\phi_{12})\,. \end{aligned} \tag{14}$$

Again, the effective mass characterizing neutrinoless double beta decay automatically looks straightforward:

$$\langle m\rangle=\left|c_{12}^2c_{13}^2c_{14}^2c_{15}^2\,m_1+s_{12}^2c_{13}^2c_{14}^2c_{15}^2\,\mathrm{e}^{2i\phi_{12}}+s_{13}^2c_{14}^2c_{15}^2\,m_3\,\mathrm{e}^{2i\phi_{13}}+s_{14}^2c_{15}^2\,m_4\,\mathrm{e}^{2i\phi_{14}}+s_{15}^2\,m_5\,\mathrm{e}^{2i\phi_{15}}\right|. \tag{15}$$

involving just the four physical Majorana phases. Regarding CP violation in oscillations, there are $\begin{pmatrix}5\\3\end{pmatrix}-3=10-3=7$ different I_{ijk} combinations with $i<j<k$ and $i,j\in\{1,2,3,4,5\}$, where the subtraction of 3 stems from the cases which have both 4 and 5 in ijk:

$$I_{123},\;I_{124},\;I_{125},\;I_{134},\;I_{135},\;I_{234},\;I_{235}\,. \tag{16}$$

There also exist $\begin{pmatrix}5\\4\end{pmatrix}-3=5-3=2$ "sumrules" in analogy to Eq. (13), namely

$$I_{123}+I_{134}-I_{124}=I_{234}\text{ and }I_{123}+I_{135}-I_{125}=I_{235}\,. \tag{17}$$

Hence, at the end there are 5 physical Dirac CP violating phase combinations affecting neutrino oscillation probabilities, for instance one could choose

$$I_{123}=\phi_{12}+\phi_{23}-\phi_{13}\,, \tag{18}$$
$$I_{124}=\phi_{12}+\phi_{24}-\phi_{14}\,, \tag{19}$$
$$I_{134}=\phi_{13}+\phi_{34}-\phi_{14}\,, \tag{20}$$
$$I_{125}=\phi_{12}+\phi_{25}-\phi_{15}\,, \tag{21}$$
$$I_{135}=\phi_{13}+\phi_{35}-\phi_{15}\,. \tag{22}$$

The phase relevant for CP violation in the short-baseline oscillations $\overset{(-)}{\nu}_e\leftrightarrow\overset{(-)}{\nu}_\mu$ sector is $\Im\left\{U_{e4}^*\,U_{\mu5}^*\,U_{\mu4}\,U_{e5}\right\}\propto\sin(\phi_{14}-\phi_{15}-\phi_{24}+\phi_{25})=\sin(I_{125}-I_{124})$.

The generalization to N_s sterile neutrinos is now clear: In more mathematical language, adopted from Ref. [29], one may define the operator δ (with $\delta^2=0$) such that (cf. Eq. (13))

$$\delta I_{1234}\equiv F_{1234}^{(4)}=I_{123}+I_{134}-I_{124}-I_{234}=0\,. \tag{23}$$

For general i,j,k,l with $i<j<k<l$ one has $\delta I_{ijkl}=0$. For five active generations one has ten I_{ijk} and five $F_{ijkl}^{(4)}=0$, one of which can be expressed by the other four, for instance

$$F_{2345}^{(4)}=F_{1235}^{(4)}+F_{1345}^{(4)}-F_{1234}^{(4)}-F_{1245}^{(4)}\text{ or }\delta F_{12345}=0\,. \tag{24}$$

Therefore, the standard result of $10-(5-1)=6$ Dirac phases is obtained. If all generations were active, then the number of independent phase combinations is

$$\binom{N}{3} - \left[\binom{N}{4} - \binom{N}{5}\right] = \binom{N}{0} - \left[\binom{N}{1} - \binom{N}{2}\right] = \frac{1}{2}\,(N-1)\,(N-2)\,. \tag{25}$$

One simply subtracts the number of $\frac{1}{2}\,N_s\,(N_s-1) = \frac{1}{2}\,(N-3)\,(N-4)$ unphysical rotations from this result to obtain the already quoted $2\,N-5 = 2\,N_s+1$ Dirac phases. The first binomial in Eq. (25) is the number of ijk combinations with $i<j<k$, the second binomial is the number of sumrules between them, and the third binomial describes the linear relations existing between the sumrules. For instance, 5 active generations have $\binom{5}{3}=10$ different I_{ijk} combinations with $i<j<k$ and $i,j\in\{1,2,3,4,5\}$. A number of $\binom{5}{4}=5$ sumrules exist, which $\binom{5}{5}=1$ linear relation between them.

III. FINAL REMARKS

In summary, the issue of a proper parametrization scheme for the lepton mixing matrix will become relevant as experiments reach sensitivity to CP violation, either of Dirac or Majorana type. Even more so, if present indications for sterile neutrinos are confirmed in upcoming experiments. While the form given in the PDG is just a re-writing of the symmetrical form proposed in Ref. [1], we have advocated here the conceptual advantages of the original symmetrical presentation for the description of neutrino oscillations and, especially, of lepton number violating processes like neutrinoless double beta decay.

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